# Kinematics of Rendezvous Maneuvers

D.L. Jensen*
*Rockwell International, Downey, California*

**Mathematical expressions are obtained that solve the rendezvous problem with evasive and passive objects. Two simple rendezvous guidance schemes are imposed on the kinematic equations of motion. Complementary solutions to the resulting differential equations are solutions for rendezvous with passive objects, and particular solutions represent rendezvous with an evasive object. The solutions depend on a nondimensional guidance effectiveness parameter. Effective evasive maneuvers are discussed and analytical results are compared with numerically computed solutions to example rendezvous cases.**

## Nomenclature and Coordinate System

$a$ = acceleration, fps$^2$
$C_e$ = first guidance effectiveness parameter, nondimensional
$C_E$ = second guidance effectiveness parameter, nondimensional
$C_\sigma$ = first guidance constant
$C_v$ = second guidance constant
$F$ = force, lb
$m$ = mass, slugs
$R$ = polar coordinate, range, ft
$t$ = time, s
$V$ = velocity, ft/s
$\lambda$ = thrust direction, deg
$\Sigma$ = evasive maneuver direction, deg or rad
$\sigma$ = polar coordinate, direction, deg or rad

*Subscripts*

$0$ = initial value
$I$ = rendezvous object
$c$ = commanded or constant value
$N$ = normal to line of sight
$R$ = along line of sight, or at rendezvous
$T$ = rendezvous object
$(\cdot)$ = derivative with respect to time

## Introduction

RENDEZVOUS maneuvers are analyzed and analytical solutions to the characteristic equations are derived which give considerable insight into the kinematics and requirements of rendezvous. The maneuvering acceleration and the impulsive velocity necessary to rendezvous determine the rendezvous vehicle's required propulsion force and the amount of fuel required to complete a rendezvous maneuver. The analysis is directed toward determination of these basic requirements.

The polar coordinate system is shown in Fig. 1. The equations of motion for this system are

$$(a_T - a_I)_N = (R\ddot{\sigma}) + (2\dot{R}\dot{\sigma}) \tag{1}$$

$$(a_T - a_I)_R = \ddot{R} - R\dot{\sigma}^2 \tag{2}$$

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*Member of Technical Staff, Control Systems Engineering. Member AIAA.

$$(V_T - V_I)_R = \dot{R} \tag{3}$$

$$(V_T - V_I)_N = (R\dot{\sigma}) \tag{4}$$

Rendezvous is the simultaneous matching of *both* the position and velocity vectors of a rendezvous vehicle and a rendezvous target. The results of the analysis are applicable to general rendezvous problems such as satellite inspection or retrieval missions in planetary orbits or possibly planetary landings. However, the results also apply to rendezvous with evasive as well as passive objects. Consequently, effective evasive maneuvers are considered and computational results are presented to support the conclusions.

## Assumptions

The kinematics considered for analysis comprise a smooth continuous maneuver that ultimately results in a successful rendezvous. When the rendezvous mode is begun, corrections are made to establish and maintain a collision course; in addition, the range rate is progressively reduced as the range decreases until both approach zero. The initial value of the range rate or relative velocity along the line of sight will always be negative. This is a practical limitation because a rendezvous vehicle should not be moving away from an intended rendezvous object in a realistic case.

It is assumed that the rendezvous vehicle responds perfectly to commands and that there is no error or delay involved in obtaining guidance information such as the angular line of sight rate, the range to the rendezvous object, or the range rate. The rendezvous vehicle is assumed to have continuously variable propulsive maneuver capability.

## Evasive Maneuvers

A rendezvous object might maneuver to avoid rendezvous. Therefore, a realistic evasive maneuver needs to be considered. If the rendezvous object detects a pursuing vehicle, it will know the direction of the pursuing vehicle and may apply an evasive maneuvering force in any direction to evade the pursuer.

In general, an evasive maneuver will have a component along the line of sight between the two vehicles and a component normal to the line of sight. The objective of the rendezvous vehicle's guidance will be to deliver the rendezvous vehicle on a collision course, i.e., a negative or closing rate rate will exist with the near zero rotation rate of the line of sight. In such a situation, an evasive maneuver along the line of sight toward the pursuer will only hasten a collision or rendezvous that the rendezvous object presumably wishes to avoid; the maneuver is therefore not useful. It appears a maneuver along the line of sight should be directed away from the rendezvous vehicle. If this maneuver is to work, the rendezvous target will attempt to establish and maintain a positive range rate before the range becomes zero. In this case, the rendezvous target's acceleration must exceed the rendezvous vehicle's acceleration.

Similarly, in this case, the velocity input of the rendezvous target will exceed the pursuer's impulsive velocity input by at least the initial range rate. The target vehicle will be playing into the rendezvous vehicle's "hand" by taking out a velocity component the pursuing vehicle would otherwise be required to remove in order to rendezvous. Therefore, a maneuver force directed along the line of sight away from the rendezvous vehicle is also concluded to be a poor choice for the target vehicle since, to be effective, it is expensive in terms of the acceleration and velocity required. If evasion along the line of sight appears to be undesirable, the remaining direction to consider for application of an evasive maneuver is normal to the line of sight at 90 to 270 deg. As before, the rendezvous vehicle will strive to make the minimum approach range as large as possible. At any instant, the estimated miss for two objects approaching one another is

$$M = -V_N R/\dot{R} = -R^2\dot{\sigma}/\dot{R} \tag{5}$$

where $M$ is the miss distance in feet and $V_N$ the relative velocity normal to the line of sight in feet per second.

By accelerating normal to the line of sight, the objective vehicle is striving to increase the predicted miss distance by increasing the velocity $V_N$. This is the opposite of what the rendezvous vehicle is trying to do and, therefore, appears to be a good philosophy to adopt for an evasive maneuver. The interceptor impulsive velocity input will be much greater than the target's input and, as shown from interceptor studies, such a maneuver requires the attacker to exceed the target maneuver acceleration in order to intercept. The rate at which fuel is expended will depend on the acceleration level deemed to be the most effective for evasion. If the fuel is spent before the rendezvous vehicle uses up its fuel, the target becomes passive.

## Rendezvous Guidance

The principle of proportional guidance is readily applicable to rendezvous vehicles.[1] The corrective maneuvering of the rendezvous vehicle is made proportional to the line-of-sight rate or possibly the relative velocity normal to the line of sight between the target and interceptor. This type of guidance will insure that the two vehicle approach a collision course, which is one of the requirements for rendezvous. However, this is only part of the rendezvous problem since the relative velocity must be zero when the two vehicles meet. The maneuvering force then must be a resultant made up of a component along the line of sight that reduces the relative closing velocity to zero as the two vehicles approach one another and a component normal to the line of sight that will reduce the line of sight rate to zero. In symbols, this is

$$a_I = (a_{IR}^2 + a_{IN}^2)^{1/2} \tag{6}$$

where acceleration along the line of sight is a function of range and range rate,

$$a_{IR} \propto (R, \dot{R}) \tag{7}$$

and acceleration normal to the line of sight is proportional to the line of sight rate or velocity,

$$a_{IN} \propto (\dot{\sigma}) \tag{8}$$

If the rendezvous vehicle is initially on a collision course with the rendezvous target, a perfect rendezvous can be made by applying a constant acceleration. Assuming a constant acceleration range closure,

$$R = a_c \tag{9}$$

$$R = R_0 + a_c t \tag{10}$$

$$R = R_0 + R_0 t + a_c t^2/2 \tag{11}$$

At the time of rendezvous $t_R$, both the range and range rate are zero. From Eqs. (10) and (11), the time to rendezvous and the necessary acceleration are, respectively,

$$t_R = -2R_0/\dot{R}_0 \tag{12}$$

and

$$a_c = \dot{R}_0^2/2R_0 \tag{13}$$

Also, from Eqs. (10) and (11), expressions for time and range rate as functions of range are obtained,

$$t = -2(R_0/\dot{R}_0)\,[1 - \sqrt{(R/R_0)}\,] \tag{14}$$

$$\dot{R} = \dot{R}_0\sqrt{(R/R_0)} \tag{15}$$

Equation (15) indicates the basis for rendezvous guidance schemes that command maneuver forces or accelerations along the line of sight proportional to the range rate and the square root of range

$$F_R = -K_R\sqrt{R} - K_{\dot{R}}\dot{R} \tag{16}$$

Such systems strive to maintain constant acceleration range closure during rendezvous.[2,3]

## Rendezvous Solutions

A solution to the two-dimensional kinematic equations (1) and (2) can be obtained by imposing conditions consistent with the rendezvous guidance requirements. Referring to Eqs. (1) and (2) and assuming the rendezvous range closing function is accomplished by constant acceleration $\ddot{R} = a_c$, then the rendezvous vehicle acceleration along the line of sight is

$$A_{IR} = -a_c + a_{TR} + R\dot{\sigma}^2 \tag{17}$$

The effect of rotation of the line of sight is to reduce the acceleration required of the rendezvous vehicle. Therefore, it is conservative to require

$$a_{IR} = -a_c + a_{TR} \tag{18}$$

Note that even though evasive maneuvers along the line of sight $a_{TR}$ are considered poor choices, the possibility of such maneuvers is not excluded by the assumption of constant range acceleration. By imposing constant acceleration range closure, the range and range rate become known functions of time. Therefore, Eq. (2) can be solved if we impose some relationship between the acceleration normal to the line of sight and the line-of-sight rate. If we try as a first case,

$$a_{IN} = C_\sigma\dot{\sigma} \tag{19}$$

then the derivative with respect to time is

$$\dot{a}_{IN} = C_{\dot{\sigma}}\ddot{\sigma} \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (1) and rearranging results in

$$(\dot{a}_{IN}R/C_\sigma) + a_{IN}\,[1 + (2\dot{R}/C_\sigma)\,] = a_{TN} \tag{21}$$

Equation (21) is a linear first-order differential equation with nonconstant coefficients. The complementary solution

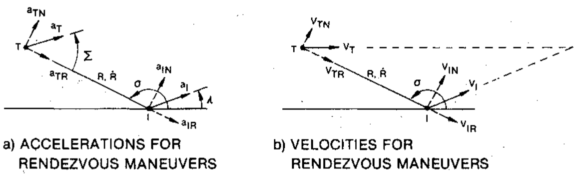



**Fig. 1 Polar coordinates for rendezvous kinematics.**

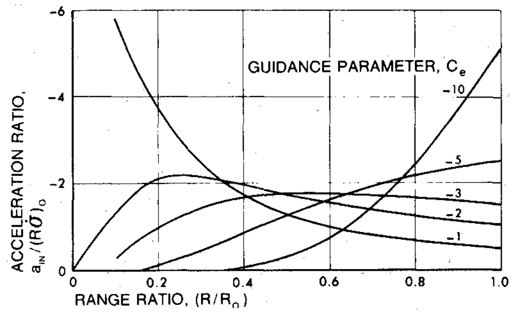



**Fig. 2 Acceleration normal to line of sight during rendezvous with passive object.**

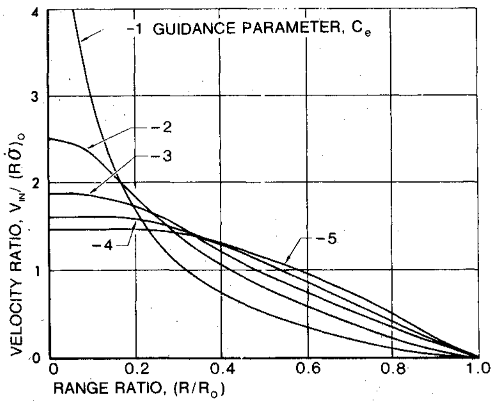



**Fig. 3 Impulsive velocity normal to line of sight during rendezvous with passive object.**

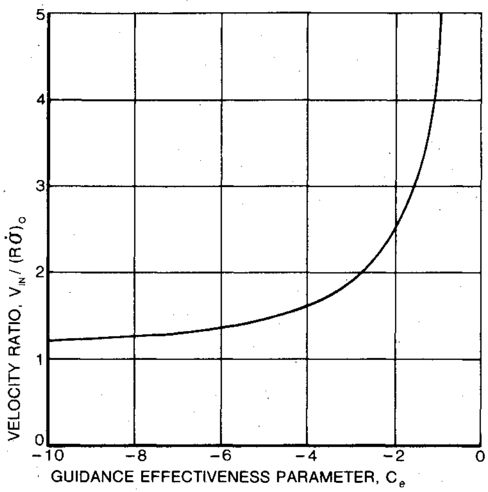



**Fig 4 Impulsive velocity normal to line of sight vs guidance effectiveness to rendevous with passive object.**

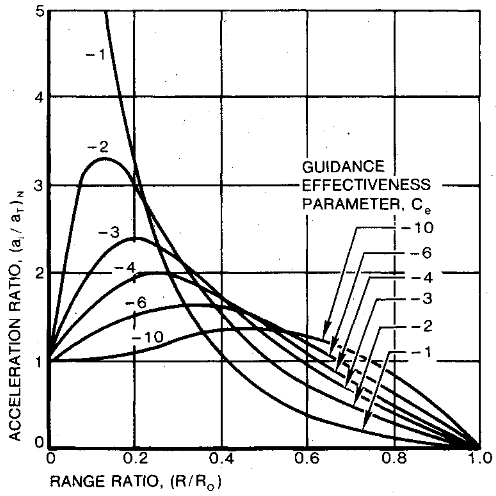



**Fig. 5 Acceleration normal to line of sight during rendezvous with evasive object.**

yields the maneuvering required normal to the line of sight to a rendezvous with a nonevasive target, while the particular solution provides similar results for an evasive target. Derivation of the solutions is included in the Appendix.

The complementary solution to Eq. (21) in nondimensional form as a function of range is

$$a_{IN}/(\dot{R}\dot{\sigma})_0=(C_e/2)(R_0/R)^2e^u \tag{22}$$

where $u=-C_e(1-\sqrt{R_0/R})$ and a guidance effectiveness parameter is defined as $C_e=2C_\sigma/\dot{R}_0$.

Equation (22) is plotted in Fig. 2. For the acceleration to remain bounded, the effective gain must be less than $-1$. The effective gain is negative because of the sign of the initial range rate. Values of effective gain between $-2$ and $-5$ are most desirable, since values greater than $-2$ will require large terminal accelerations, while values less than $-5$ require extremely large initial accelerations. Integration of Eq. (22) yields the velocity input normal to the line of sight. The result in nondimensional form is given by Eq. (23),

$$V_{IN}/(R\dot{\sigma})_0=C_e^2-2C_e+2-[(C_e\sqrt{R_0/R})^2-2(C_e\sqrt{R_0/R})+2]e^u/C_e^2 \tag{23}$$

The total velocity input to rendezvous is obtained by evaluating Eq. (23) at range zero

$$V_{IN}/(R\dot{\sigma})_0=1-2/C_e+2/C_e^2 \tag{24}$$

As the absolute value of the effective guidance parameter approaches infinity, the input velocity ratio approaches one. Equations (23) is plotted in Fig. 3 and Eq. (24) in Fig. 4.

If the target maneuver normal to the line of sight consists of a constant acceleration, the particular solution for Eq. (21) is

$$(a_I/a_T)_N=(1/6)\{V^3-V^2+2V+V^4e^V[LN(-V)+\Sigma(-V)^n/n\cdot n!-(1/C_e-1/C_e^2+2/C_e^3)e^{-C_e}-LN(-C_e)-\Sigma(-C_e^n/n\cdot n!)]\} \tag{25}$$

It can be shown that the limit of this expression as range approaches zero is one, as shown in Fig. 5. As in the case of the nonevasive target, an effective gain less than $-2$ is required for the acceleration to remain finite.

Integration of Eq. (25) to obtain the velocity input necessary for rendezvous gives

$$(V_I/V_T)_N=-(C_e/6)\{(V^2-2V+2)e^V[(LN|V|+\Sigma_n(-V)^n/n\cdot n!-C_I]+V\}+C_V \tag{26}$$

where

$$V = C_e\sqrt{R_0/R}$$

$$C_V = (C_e/6)\,[\,(C_e^2 - 2C_e + 2)\,(-1/C_e + 1/C_e^2 - 2/C_e^3) + C_e\,]$$

$$C_I = (1/C_e - 1/C_e^2 + 2/C_e^3)\,e^{-C_e} + LN\,|C_e| + \Sigma_n(-C_e)^n/n\cdot n!$$

The evaluation of Eq. (26) as the range approaches zero determines the total velocity input normal to the line of sight. In the limit Eq. (26) becomes

$$(V_I/V_T)_N = 1 - 1/C_e + 2/3C_e^2 \qquad (27)$$

showing the decrease in the velocity ratio with the increasing guidance effectiveness. Equation (27) is plotted in Fig. 6.

An alternative guidance scheme was also investigated where the normal acceleration is made proportional to the relative velocity normal to the line of sight,

$$a_{IN} = C_V(R\dot{\sigma}) \qquad (28)$$

Taking the derivative

$$\dot{a}_{IN} = C_V R\ddot{\sigma} + C_V\dot{R}\dot{\sigma} \qquad (29)$$

and substituting Eqs. (28) and (29) into Eq. (1), the resulting differential equation to be solved is

$$\dot{a}_{IN}/C_V + a_{IN}\,[\,1 + (\dot{R}/C_V R)\,] = a_{TN} \qquad (30)$$

The complementary solution is

$$a_{IN}/(\dot{R}\dot{\sigma})_0 = (C_E/2)\,(R_0/R)\cdot e^V \qquad (31)$$

where

$$C_E = 2C_V R_0/\dot{R}_0, \qquad V = C_E(1 - \sqrt{R/R_0}) \qquad (32)$$

Equation (31) is plotted in Fig. 7. In this case, there is no value of guidance effectiveness that will result in finite accelerations as the range approaches zero. Integrating Eq. (31) gives the velocity input normal to the line of sight necessary to rendezvous,

$$V_{IN}/(R\dot{\sigma})_0 = C_E^2\cdot\exp(C_E)\cdot\{\,LN\sqrt{R_0/R} - 1/(C_E\sqrt{R/R_0})$$
$$\times\exp(C_E\sqrt{R/R_0}) - \Sigma_n(-C_E\sqrt{R/R_0})^n/n\cdot n!$$
$$+ \Sigma_n(-C_E)^n/n\cdot n! + [\,1/C_E\exp(C_E)\,]\,\} \qquad (33)$$

Equation (33) is evaluated at range zero to obtain the total velocity input

$$V_{IN}/(R\dot{\sigma})_0 = C_E^2\exp(C_E)\,[\,LN\sqrt{(R_0/R)}$$
$$+ \Sigma_n(-C_E)^n/n\cdot n! - \sqrt{(R_0/R)}/C_E\,] + C_E \qquad (34)$$

This expression becomes infinite at zero range, but remains finite for very small range. The expression is plotted in Fig. 8 for a value of range ratio $R/R_0 = 0.0001$, which corresponds to approximately 30-ft range if the rendezvous maneuver began at 50 miles.

The requirements against an evasive target are obtained from the particular solution to Eq. (30). Expressions for the acceleration and velocity requirements are

$$(a_I/a_T)_N = 1 - 2\sqrt{(R_0/R)}/C_E + 2(R_0/R)/(C_E^2)$$
$$-\,[\,1 - (2/C_E) + (2/C_E^2)\,]\,(R_0/R)\,(e^V) \qquad (35)$$

and

$$(V_I/V_T)_N = (1/C_E)\,\{\,(C_E^2 - 2C_E + 2)\,(e^{C_E})\,[\,(1/C_E e^{C_E})$$
$$-\sqrt{(R_0/R)}/C_E + LN(R_0/R) + \Sigma_n(-C_E)^n/n\cdot n!\,]$$
$$+ C_E + 2\sqrt{(R_0/R)}/C_E - 2LN(R_0/R)\,\} \qquad (36)$$

Equations (35) and (36) are plotted in Figs. 8 and 9. Both equations become infinite as the range approaches zero. The

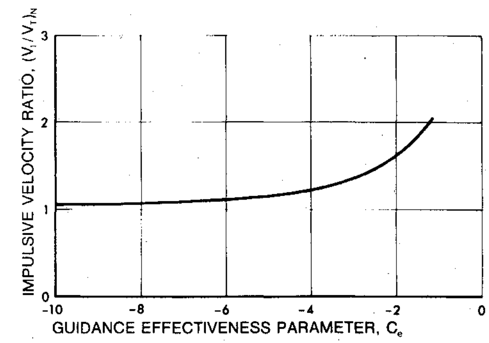



**Fig. 6 Impulsive velocity ratio normal to line of sight vs guidance effectiveness to rendezvous with evasive object.**

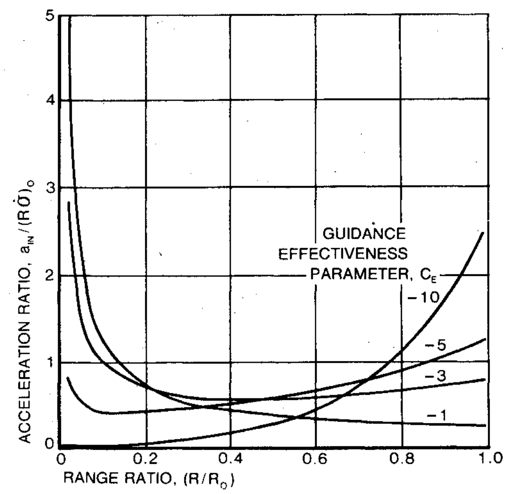



**Fig. 7 Acceleration normal to line of sight during rendezvous with passive object.**

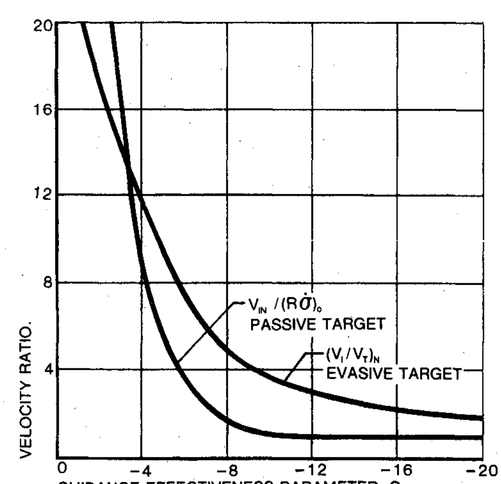



**Fig. 8 Impulsive velocity required for rendezvous vs guidance effectiveness.**

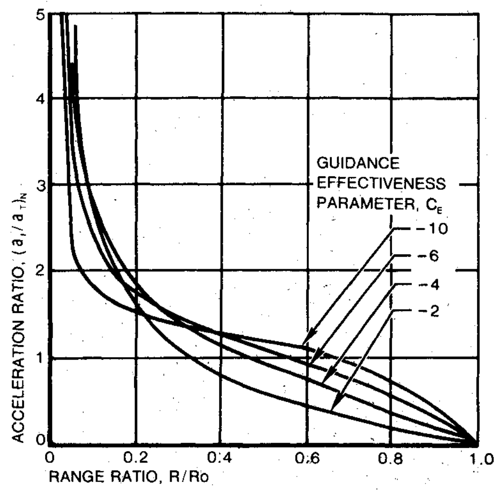



**Fig. 9 Acceleration normal to line of sight to rendezvous with evasive object.**

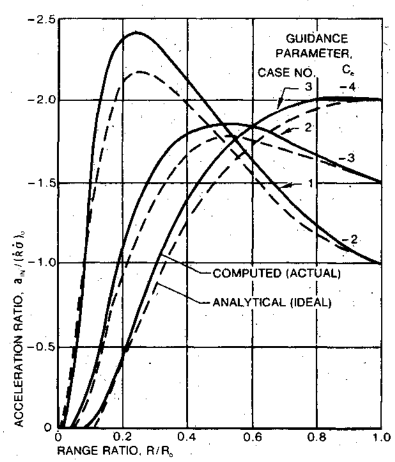



**Fig. 10 Comparison of analytical and computed rendezvous maneuvers with passive objects.**

velocity ratio is therefore evaluated for $R/R_0 = 0.001$ as before.

## Calculated Solutions and Comparisons with Analysis

A digital computer was used to provide supporting calculations for the rendezvous analysis. A two-dimensional rendezvous was programmed. A throttleable (variable thrust) rocket motor was incorporated, but time delays or other system dynamics were not considered. Equations (1) and (2) were used for the dynamics with the addition of terms to account for gravitational effects.[4]

Rendezvous guidance was accomplished by commanding thrust as a function of range, range rate, and the angular rate of the line of sight between the target and the rendezvous vehicle. The equations used to accomplish rendezvous guidance are as follows.

Total rendezvous maneuver force is

$$F_c = (F_R^2 + F_N^2)^{1/2}, \text{ lb} \tag{37}$$

where the maneuver force along the line of sight is

$$F_R = -k_R\sqrt{R} - k_{\dot R}\dot R, \text{ lb} \tag{38}$$

and the maneuver force normal to the line of sight is

$$F_n = k_\sigma \dot\sigma, \text{ lb} \tag{39}$$

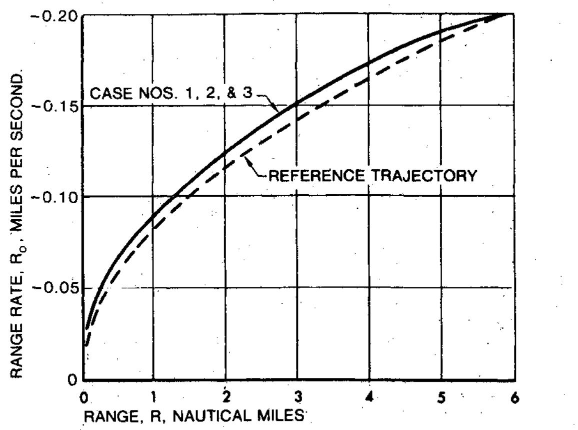



**Fig. 11 Comparison of computed trajectories with ideal reference trajectory during rendezvous with passive objects.**

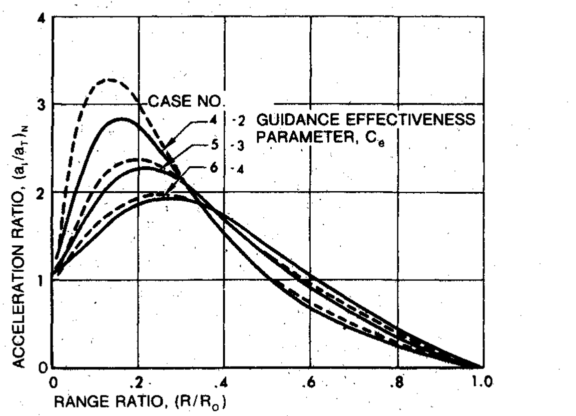



**Fig. 12 Comparison of analytical and calculated accelerations during rendezvous with evasive objects.**

The maneuver force direction was determined by

$$\lambda = \sigma + \tan^{-1}(F_N/F_R) \text{ deg} \tag{40}$$

The guidance sensitivity factors have dimensions and values of

$$k_\sigma = \pi(mR)_0 C_e / 2(180) \text{ lb/deg/s}$$

$$k_R = 50{,}000 \text{ lb/(nm)}^{1/2}$$

$$k_{\dot R} = 612{,}000 \text{ lb/nm/s}$$

A series of rendezvous were calculated for comparison with the analysis. Table 1 lists the conditions for these cases. Cases 1-6 were compared with the analytical rendezvous solutions and cases 7-11 substantiated the conclusions regarding the effectiveness of evasive maneuvers.

Figures 10-13 are plots of cases 1-6 and show substantial agreement with the results predicted from the analysis. Figures 10 and 11 show rendezvous histories against nonevasive targets and Figs. 12 and 13 the results of rendezvous with an evasive target when the target evades by maneuvering continuously normal to the line of sight. The phase plane plots of range rate vs range (Figs. 11 and 13) show that a constant acceleration range closure trajectory is followed very closely. Several reasons exist for the deviations that appear: 1) the mass changes of the rendezvous vehicle effctively provided increases in guidance effectiveness during the rendezvous so that the effective guidance parameter was not constant; 2) the deviations from the constant acceleration reference trajectory that do occur depart from the assumption of a perfect closure used in the analysis; 3) the effect of

**Table 1 Tabulation of computed rendezvous**

| Case no. | $R_0$, nm | $\dot{R}_0$, nm/s | $(R\dot{\sigma})_0$, nm/s | $C_e$ | $a_{c_2}$, fps | $(a_{IN})_0$ g | $\Sigma$ deg | $-2R_0/\dot{R}_0$, s |
|---|---|---|---|---|---|---|---|---|
| 1 | 6.0 | −0.2 | 0.033 | −2 | 20.3 | 0 | – | 60 |
| 2 | 6.0 | −0.2 | 0.033 | −3 | 20.3 | 0 | – | 60 |
| 3 | 6.0 | −0.2 | 0.033 | −4 | 20.3 | 0 | – | 60 |
| 4 | 6.0 | −0.2 | 0 | −2 | 20.3 | 1.0 | 90 | 60 |
| 5 | 6.0 | −0.2 | 0 | −3 | 20.3 | 1.0 | 90 | 60 |
| 6 | 6.0 | −0.2 | 0 | −4 | 20.3 | 1.0 | 90 | 60 |
| 7 | 6.0 | −0.2 | 0 | −3 | 20.3 | 1.0 | 0 | 60 |
| 8 | 6.0 | −0.2 | 0 | −3 | 20.3 | 1.0 | 45 | 60 |
| 9 | 6.0 | −0.2 | 0 | −3 | 20.3 | 1.0 | 90 | 60 |
| 10 | 6.0 | −0.2 | 0 | −3 | 20.3 | 1.0 | 135 | 60 |
| 11 | 6.0 | −0.2 | 0 | −3 | 20.3 | 1.0 | 180 | 60 |

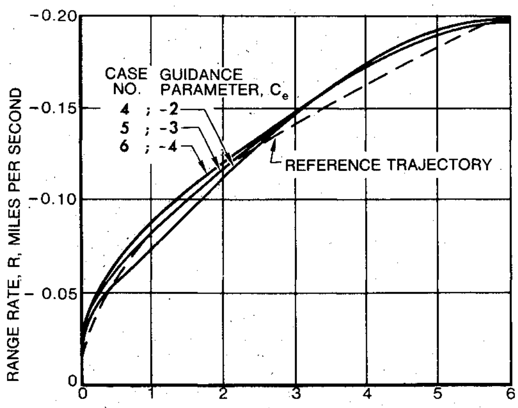
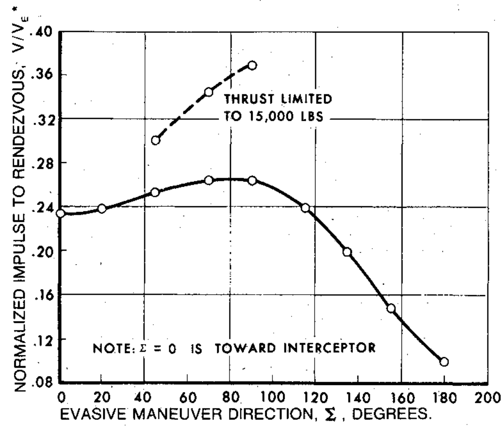



**Fig. 13 Comparison of computed trajectories with ideal reference trajectories during rendezvous with evasive objects.**




* ROCKET EXHAUST VELOCITY, 10,000 FT/SEC

**Fig. 15 Effect of evasive maneuver direction on rendezvous impulse.**

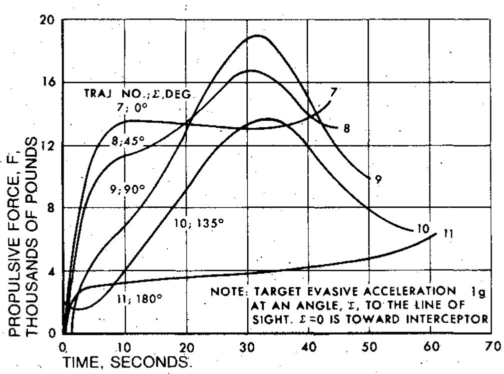



**Fig. 14 Propulsive force histories during rendezvous for various directions of evasive maneuvers.**

gravity was included in the computed results[4]; and 5) in the evasive cases, the target did not maneuver at a constant acceleration during the rendezvous, but continually increased its acceleration due to the decreasing mass with propellant use.

The effectiveness of the evasive maneuver direction is shown in Figs. 14 and 15, which are plots of the computed cases 7-11. The nominal effective guidance parameter for all of these cases is $C_e = -3$, which was chosen as a representative value on the basis of the results of a stability analysis. The results are in substantial agreement with the conclusions reached during the discussion of evasive maneuvers. The case where the target maneuver is normal to the line of sight, $\Sigma = 90$ deg, is the most expensive to the rendezvous vehicle in terms of fuel consumed, and is also the case that requires a higher acceleration level over a longer period of time. Figure 14 shows thrust histories required against evasive targets where the direction of the evasive force $\Sigma$ is a parameter. Evasion normal to the line of sight is seen to require the highest thrust level. Figure 15 shows the total velocity input and the time to rendezvous as functions of the direction of evasion. The maximum velocity input occurs for an evasive maneuver normal to the line of sight. However, the required velocity changes slowly with the direction of the evasive maneuver, so that the target would need only crude pointing information and accuracy to execute the near optimum evasive maneuver. Figure 15 also shows the effect of a maneuver-limited rendezvous vehicle. These are represented by the dashed line data where the rendezvous vehicle's maneuvering force was limited to 15,000 lb. The increase in propellant requirements is significant, amounting to a 50% increase at $\Sigma = 90$ deg if sufficient maneuver force is not available.

## Conclusions

The study indicates that rendezvous guidance utilizing the line-of-sight rate for steering is superior to one utilizing relative velocity normal to the line of sight. This is because the maneuver accelerations remain finite in the former case. It is also concluded that rendezvous evasive maneuvers are most effective when directed normal to the line of sight. The analytical solutions obtained are useful for determining rendezvous vehicle propulsion requirements and have been substantiated by independent computations. A guidance effectiveness parameters of three or greater is necessary for effective rendezvous maneuvers. It is also important that a rendezvous vehicle not be maneuver limited, but have a capability meeting that required by the rendezvous guidance. Otherwise, excessive propellants will be consumed and rendezvous might not be achieved.

## Appendix: Solution of the Differential Equations for Rendezvous

The differential equation

$$(\dot{a}_{IN}R/C_\sigma)+a_{IN}[1-(2\dot{R}/C_\sigma)]=a_{TN} \tag{21}$$

is a linear differential equation with nonconstant coefficients. Variables are separable to obtain the complementary solution $Y_c$, i.e., when $a_{TN}\equiv 0$.

The result is

$$\mathrm{d}(a_{IN})/a_{IN}=-(C_\sigma+2\dot{R})\mathrm{d}t/R$$

Substitute

$$\dot{R}=\dot{R}_0+a_c t \quad \text{and} \quad R=R_0+\dot{R}_0 t+a_c t^2/2$$

and integrating

$$LN(a_{IN})=-\int[(C_\sigma)+2(\dot{R}_0+a_c t)]\mathrm{d}t/$$
$$(R_0+\dot{R}_0 t+a_c t^2/2)+C_1$$

Evaluating the integrals

$$-(2R_0+C_\sigma)\int \mathrm{d}t/R=2(2\dot{R}_0+C_\sigma)/(\dot{R}_0+a_c t)$$

and

$$-2a_c\int t\mathrm{d}t/R=-2LN(R)-4\dot{R}_0/(\dot{R}_0+a_c t)$$

Therefore,

$$LN(a_{IN})=2(C_\sigma)/(\dot{R}_0+a_c t)-2LN(R)+C_1$$

Taking the antilog,

$$a_{IN}R^2=C_0e^v$$

where $C_0=e^{C_1}$ and $V=2C_\sigma/(\dot{R}_0+a_c t)$.

The complementary solution is

$$a_{IN}=C_0e^v/R^2$$

At $T=0$, $R=R_0$, and $a_{IN}=C_0e^{C_e}/R_0^2=C_\sigma\dot{\sigma}_0$; $C_e=2C_\sigma/\dot{R}_0$.

Therefore,

$$C_0=C_\sigma\dot{\sigma}_0R_0^2/e^{C_e} \quad \text{and} \quad (a_{IN}/\dot{R}_{\dot{\sigma}}\sigma_0)=(C_e/2)(R_0/R)^2\cdot e^u$$

where $u=-C_e(1-\sqrt{R_0/R})$ is the complementary solution in nondimensional form.

To determine a particular solution,

$$(a_T-a_I)_N=(2\dot{R}a_{IN}+R\dot{a}_{IN})/C_\sigma$$

use $\dot{R}=\dot{R}_0\sqrt{R/R_0}$.

Then

$$(2\dot{R}/C_\sigma)=(2\dot{R}_0\sqrt{R/R_0}/C_\sigma)=4\sqrt{R/R_0}/C_e=4/V$$

where $V=C_e\sqrt{R_0/R}$ and $C_e=C_\sigma/\dot{R}_0$.

Substituting for $\mathrm{d}V/\mathrm{d}t$ and $\mathrm{d}(a_{IN})/\mathrm{d}V$ in the differential equation and rearranging,

$$[\mathrm{d}(a_{IN})/\mathrm{d}V]-a_{IN}(1+4/V)=-a_{TN}$$

This is a standard form that can be solved by obtaining an integrating factor. Let

$$P(V)=-(1+4/V)$$

Then $\exp p(V)\mathrm{d}V=e^{-V}/V^4$.

Multipying both sides of the differential equation by the integrating factor

$$\{[\mathrm{d}(a_{IN})/\mathrm{d}V]-a_{IN}(1+4/V)\}(e^{-V}/V^4)$$
$$=-a_{TN}(e^{-V}/V^4)$$

and integrating,

$$a_{IN}(e^{-V}/V^4)=-a_{TN}(e^{-V}/V^4)\mathrm{d}V+C1$$

and

$$\int(e^{-V}/V^4)\mathrm{d}V=-(e^{-V}/3V^3)+(e^{-V}/6V^2)$$
$$+(e^{-V}/6V)+LN(V)/6+(1/6)\Sigma(-V)^n/n\cdot n!$$

leads, finally, to

$$a_{IN}/a_{TN}=(1/6)\{2V-V^2+V^3+V^4e^v[LN(V)$$
$$+\Sigma(-V)^n/n\cdot n!]\}+C1\cdot V^4e^V/a_{TN}$$

when $V=C_e$, $a_{IN}=0$, and

$$C1=-(a_{TN}e^{-C_e}/6C_e^4)(2C_e-C_e^2+C_e^3)$$
$$-(a_{TN}/6)[LN(|C_e|)+\Sigma(-C_e)^n/n\cdot n!]$$

The total solution is the sum of the complementary and the particular integal. The velocity is obtained by integration,

$$V_{IN}=\int a_{IN}\mathrm{d}t$$

The second possibility investigated for rendezvous guidance was to make the correction maneuver proportional to the relative velocity normal to the line of sight,

$$a_{IN}=C_VR\dot{\sigma}$$

As has been previously shown, the resulting differential equation is

$$(\dot{a}_{IN}/C_V)+a_{IN}[1+(\dot{R}/R)/C_V)]=a_{TN} \tag{30}$$

To obtain the homogeneous solution, separate the variables and set the right-hand side equal to zero,

$$[1+(\dot{R}/C_VR)]C_V\mathrm{d}t=-\mathrm{d}a_{IN}/a_{IN}$$

Integrating

$$LN(a_{IN}/C1)=-(180C_Vt/\pi)-\int(\dot{R}/R)\mathrm{d}t$$

By definition, $\dot{R}=\dot{R}_0+a_ct$ and $R=R_0+\dot{R}_0t+a_ct^2/2$.
Therefore,

$$LN(a_{IN}/C1)=-(180C_Vt/\pi)-LN(R)$$

Taking the antilog of both sides,

$$a_{IN}R/C1=e^{-W}$$

where $W=C_Vt$.

Now

$$a_{IN}=(C1/R)e^{-W}=C_VR_0\dot{\sigma}_0 \quad \text{at} \quad t=0$$

Therefore,

$$C1=C_VR_0^2\dot{\sigma}_0$$

Substitute

$$t=-(2R_0/\dot{R}_0)[1-\sqrt{R/R_0}] \quad \text{and} \quad C_E=C_V R_0/\dot{R}_0$$

Then

$$(a_{IN}/\dot{R}_0\dot{\sigma}_0)=(C_E/2)(R_0/R)\cdot e^{-Z}$$

where

$$Z=C_E(1-\sqrt{R/R_0})$$

For a particular integral $Y_P$, assume

$$Y_P=\lambda_1+\lambda_2 Y_C$$

where $Y_c$ is complementary solution.
Then

$$\dot{Y}_P=\dot{\lambda}_1+\dot{\lambda}_2 Y_c+\lambda_2\dot{Y}_0$$

Let

$$\dot{\lambda}_1+\dot{\lambda}_2 Y_c\equiv 0$$

Substituting in the differential equation

$$(\lambda_2/C_V)(\mathrm{d}Y_c/\mathrm{d}t)+(\lambda_1+\lambda_2 Y_c)[1+(\dot{R}/C_V R)]=a_{TN}$$

let

$$\lambda_1=a_{TN}/[1+(2/C_E\sqrt{R/R_0})]$$

By differentiation

$$\dot{\lambda}_1=a_{TN}C_E\dot{R}_0/R_0(2+C_E\sqrt{R/R_0})^2$$

Now,

$$\dot{\lambda}_2=-\dot{\lambda}_1/Y_c=-(a_{TN}/R_0\dot{\sigma}_0\cdot e^{C_E})(R/R_0\cdot e^{C_E\sqrt{R/R_0}})/$$
$$(2+C_E\sqrt{R/R_0})^2$$

Let $V=C_E\sqrt{R/R_0}$; then $\dot{\lambda}_2=A\int(V^2e^V\mathrm{d}t)/(V+2)^2+C2$,

where $A=-360\,a_{TN}/(R_0\dot{\sigma}_0C_E^2\cdot e^{C_E})$.

Therefore, by appropriate substitition,

$$\lambda_2=(2R_0A/C_E\dot{R}_0)\int(V^2e^V\mathrm{d}V)/(V+2)^2+C2$$

Let $U=V+2$, and substituting

$$\lambda_2=(2R_0Ae^{-2}/C_E\dot{R}_0)$$
$$\times[\int e^U\mathrm{d}U-4\int e^U\mathrm{d}U/U+4\int e^U\mathrm{d}U/U^2+C2]$$

After integration and algebraic manipulation,

$$\lambda_2=-(4a_{TN}/\dot{R}_0\dot{\sigma}_0C_E^3)[(C_E\sqrt{R/R_0}-2)$$
$$\times\exp[-C_E(1-\sqrt{R/R_0})]/(C_E\sqrt{R/R_0}+2)]+C2$$

The results are combined $Y_P=\lambda_1+\lambda_2 Y_c$,

$$Y_P=[a_{TN}/(C_E\sqrt{R/R_0})^2]/[(C_E\sqrt{R/R_0})^2-2(C_E\sqrt{R/R_0})$$
$$+2]+(C2\cdot C_V R_0^2\dot{\sigma}_0/R)\cdot\exp[C_E(1-\sqrt{R/R_0}]$$

At $R=R_0$.

$$Y_P=(a_{TN}/C_E^2)[C_E^2-2C_E+2]+(C2\cdot C_V R_0\dot{\sigma}_0)$$

Solving for

$$C2=-a_{TN}(C_E^2-2C_E+2)/C_V R_0\dot{\sigma}_0 C_E^2$$

the final result is

$$a_{TN}/a_{TN}=(1-2V+2/V^2)-(1-2/C_E$$
$$+2/C_E^2)(R_0/R)\cdot e^{(1-V)}$$

where $V=C_E\sqrt{R/R_0}$.

Again, velocity inputs are obtained by integration,

$$V_{IN}=\int a_{IN}\mathrm{d}t$$

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